Seventh exercise sheet for the lecture

The Algebraic Path Problem

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Winter Term 2006/07

Exercise 24 (semirings)

1. Which of the following is a semiring?

$$
\begin{array}{ll}
(\mathbb{Z}, -, \cdot, 0, 1) & \\
(\mathbb{N}, \ominus, \cdot, 0, 1) & \text{where } a \ominus b = |a - b| \\
(\mathbb{Z}, +, \cdot, 0, 1) & \\
(\mathbb{Z}, \cdot, +, 1, 0) & \\
(\mathbb{Z}_\infty, \mathit{min}, +, \infty, 0) & \\
(\mathbb{N}, \mathit{max}, \cdot, 0, 1) & \\
(\mathcal{P}(\Sigma^*), \cup, \cap, \emptyset, \Sigma^*) & \\
(\mathcal{P}(\Sigma^*), \cap, \cdot, \Sigma^*, \{\varepsilon\}) & \text{where } \cdot \text{ is language concatenation}
\end{array}
$$

2. Is there a semiring in which the two operations coincide? $\diamond$

Exercise 25 (infinitary sum operations and complete semirings)

1. In which sense do the completeness conditions (C1) and (C2) for an infinitary sum operation $\sum$ generalize the commutativity and associativity of the underlying semiring addition?

2. Show that (C1) and (C2) imply the following statements:

   (a) for every $j \in I$: $\sum\limits_{i \in I} k_i = k_j + \sum\limits_{i \in I \setminus \{j\}} k_i$

   (b) for every $I' \subseteq I$: $\sum\limits_{i \in I} k_i = (\sum\limits_{i \in I'} k_i) + (\sum\limits_{i \in I \setminus I'} k_i)$

   (c) $\sum\limits_{j \in J, i \in I} k_{j,i} = \sum\limits_{j \in J} (\sum\limits_{i \in I} k_{j,i})$

3. Show that two infinitary sum operations for the same semiring must coincide on finite index sets if both fulfill the completeness conditions.

4. Show that for the boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ there is exactly one infinitary sum operation $\sum$ such that $(\mathbb{B}, \sum)$ is complete.

5. Is there a semiring K such that there are two different infinitary sum operations $\sum^1$ and $\sum^2$ with both $(K, \sum^1)$ and $(K, \sum^2)$ being complete?

6. Is there a (finite) semiring such that there is no infinitary sum operation making it complete? ◇

Exercise 26 (induced partial order, "algebraic Dijkstra")
Let $(K, +, \cdot, 0, 1)$ be a semiring.

1. Impose a suitable condition on the semiring allowing the definition of a partial order $\leq$ on K such that for every $a, b \in K$, $a + b$ is the smallest $c \in K$ with $a, b \leq c$.

2. Show that then for every $a, b, c \in K$, $a \leq b$ implies $c \cdot a \leq c \cdot b$ and $a \cdot c \leq b \cdot c$.

3. Which of the relevant semirings from the lecture and Exercise 24 fulfill the additional requirements $a + b \in \{a, b\}$ and $a \cdot b \leq a, b$ for every $a, b \in K$? ◇