Thirteenth exercise sheet for the lecture

The Algebraic Path Problem

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Exercise 43 (closure of $\boldsymbol{K^{\mathrm{rec},\varphi}\langle\langle\Delta^*\rangle\rangle}$ under $+$ and $\odot_\varphi$)
Let Δ be an alphabet, $(K,+,\cdot,0,1)$ be a semiring, $\varphi : K \to K$ be a semiring endomorphism, and M_1 and M_2 be weighted automata over Δ and K. Show that there is a weighted automaton N' over Δ and K with:

2. $S^\varphi_{N'} = S^\varphi_{M_1} \odot_\varphi S^\varphi_{M_2}$. ◇

Exercise 44 (scalar product)
Let Δ be an alphabet, $(K,+,\cdot,0,1)$ be a semiring, and $\varphi : K \to K$ be an endomorphism on it. Show that $k \cdot (S \odot_\varphi T) = (k \cdot S) \odot_\varphi T$ for every $k \in K$ and $S, T \in K\langle\langle\Delta^*\rangle\rangle$. ◇

Exercise 45 (φ-Kleene-+)
Give an equivalent definition of $(.)^{+,\varphi}$ without using $\odot_\varphi$. ◇

Exercise 46 (φ-rational formal power series)
Let Δ be an alphabet, $(K,+,\cdot,0,1)$ be a semiring, and $\varphi : K \to K$ be a semiring endomorphism. Show that the formal power series $S \in K\langle\langle\Delta^*\rangle\rangle$ defined by $(S,w) = 1$ for every $w \in \Delta^*$, is φ-rational, i.e. that $S \in K^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle$. ◇

Exercise 47 (φ-rational formal power series)
Let $\Delta = \{a,b\}$ and let $S \in \mathbb{N}_\infty\langle\langle\Delta^*\rangle\rangle$ be the formal power series defined by

$$(S,w) = \begin{cases} n & \text{if } w = a^n b^k \text{ for some } n,k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

for every $w \in \Delta^*$.

1. Is S id-rational over the semiring $(\mathbb{N}_\infty, min, +, \infty, 0)$?
2. Is S id-rational over the semiring $= (\mathbb{N}_\infty, max, \cdot, 0, 1)$? ◇

Exercise 48 (rationality implies recognizability)
Let Δ be an alphabet, $(K, +, \cdot, 0, 1)$ be a semiring, and $\varphi : K \to K$ be a semiring endomorphism. Show that $K^{\mathrm{rat},\varphi}\langle\langle\Delta^*\rangle\rangle \subseteq K^{\mathrm{rec},\varphi}\langle\langle\Delta^*\rangle\rangle$. $\diamond$