First exercise sheet for the lecture

The Algebraic Path Problem

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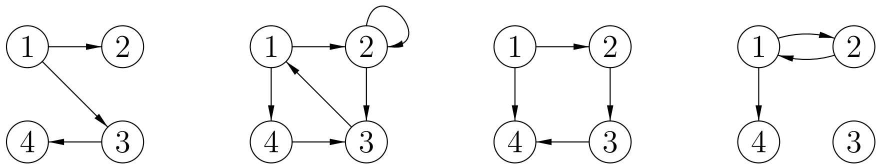
Exercise 1 (graph and path properties, path sets)

1. Which of the following (directed) graphs are cycle free? Give the sets of elementary cycles and of simple cycles for each graph that is not.



2. Give the sets of maximal elementary paths and of maximal simple paths for each of the graphs above.

3. Give the set $P_{1,4}$ for each of the graphs above.

4. Which of the following statements hold? Prove or disprove.

   (a) Every elementary path is simple.

   (b) Every simple path is elementary.

   (c) For vertices u and v, the set of simple paths from u to v is finite.

   (d) A path is simple if its inner vertices are pairwise different.

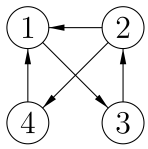
   (e) A path is elementary if its inner vertices are pairwise different.

   (f) For vertices u, v, and w, $P_{u,v} \cdot P_{v,w} = P_{u,w}$.

   (g) For vertices u, v, and w, $|P_{u,v} \cdot P_{v,w}| = |P_{u,v}| \cdot |P_{v,w}|$.

   (h) For a vertex u, $P^*_{u,u} = P_{u,u}$. $\diamond$

Exercise 2 (weighted graphs, reliability of information transfer)
Consider the following graph:



Let it be $[0, 1]$-weighted as follows, where $\lambda(e)$ is interpreted as the reliability of information transfer along edge e:

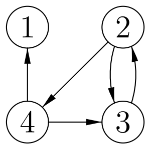
e	$(1, 3)$	$(2, 1)$	$(2, 4)$	$(3, 2)$	$(4, 1)$
$\lambda(e)$	0.5	0.4	0.7	0.2	0.6

For every pair of different vertices $u, v \in \{1, 2, 3, 4\}$, compute the highest attainable information transfer reliability between them. $\diamond$

Exercise 3 (adjacency matrix and matrix multiplication)
Let $G = (V, E)$ be a graph, where $V = \{1, \ldots, n\}$. The *adjacency matrix* A_G of G has n rows, n columns, and is defined (for $u, v \in V$) as follows:

$$[A_G]_{u,v} = \begin{cases} 1 & \text{if } (u, v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

1. Give the adjacency matrix A of the following graph.



2. Compute the matrix multiplications $A^2 = A \cdot A$ and $A^3 = A \cdot A \cdot A$. How can the entries of A^2 and A^3 be interpreted in terms of the graph?

   (Recall that for $n \times n$ matrices M and N, their multiplication is defined by $[M \cdot N]_{u,v} = \sum_{1 \leq t \leq n} [M]_{u,t} \cdot [N]_{t,v}$ for every $1 \leq u, v \leq n$.) $\diamond$