Type-Based Reasoning and Bidirectional Transformation

Janis Voigtländer

Technische Universität Dresden

February 20th, 2009

Polymorphic Types: An Example in Haskell

A standard function:

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$\texttt{map not}\ [1, 2, 3]$ $\lightning$ rejected at compile-time

Another Example

$$
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For every choice of p, f, and l:

$$\texttt{takeWhile}\ p\ (\texttt{map}\ f\ \ l) \quad = \quad \texttt{map}\ f\ (\texttt{takeWhile}\ (p \circ f)\ \ l)$$

Provable by induction.

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- $\mathrm{g} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$ must work uniformly for every instantiation of α.

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- That is what was claimed!

Automatic Generation of Free Theorems

At `http://linux.tcs.inf.tu-dresden.de/~voigt/ft`:

This tool allows to generate free theorems for sublanguages of Haskell as described here.

The source code of the underlying library and a shell-based application using it is available here and here.

Please enter a (polymorphic) type, e.g. "(a -> Bool) -> [a] -> [a]" or simply "filter":

```
g :: (a -> Bool) -> [a] -> [a]
```

Please choose a sublanguage of Haskell:

- (•) no bottoms (hence no general recursion and no selective strictness)
- ( ) general recursion but no selective strictness
- ( ) general recursion and selective strictness

Please choose a theorem style (without effect in the sublanguage with no bottoms):

- (•) equational
- ( ) inequational

[Generate]

Automatic Generation of Free Theorems

The theorem generated for functions of the type

```
g :: forall a . (a -> Bool) -> [a] -> [a]
```

in the sublanguage of Haskell with no bottoms is:

```
forall t1,t2 in TYPES, R in REL(t1,t2).
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall (x, y) in R. p x = q y)
   ==> (forall (z, v) in lift{[]}(R).
         (g p z, g q v) in lift{[]}(R))
```

The structural lifting occurring therein is defined as follows:

```
lift{[]}(R)
  = {([], [])}
  u {(x : xs, y : ys) |
       ((x, y) in R) && ((xs, ys) in lift{[]}(R))}
```

Reducing all permissible relation variables to functions yields:

```
forall t1,t2 in TYPES, f :: t1 -> t2.
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall x :: t1. p x = q (f x))
   ==> (forall y :: [t1]. map f (g p y) = g q (map f y))
```

Export as PDF   Show type instantiations   Enter a new type   Help page

DFG-Project VO 1512/1-1

Free theorems

- POPL'04
- TCS'07
- I&C'09

DFG-Project VO 1512/1-1

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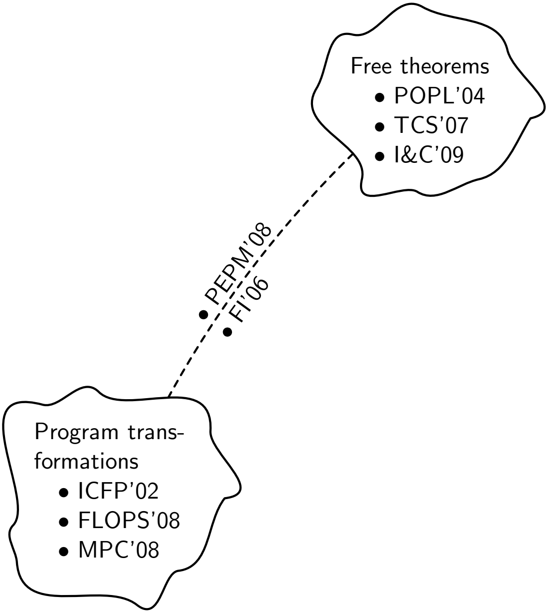
- POPL'04
- TCS'07
- I&C'09

Program transformations

- ICFP'02
- FLOPS'08
- MPC'08

DFG-Project VO 1512/1-1



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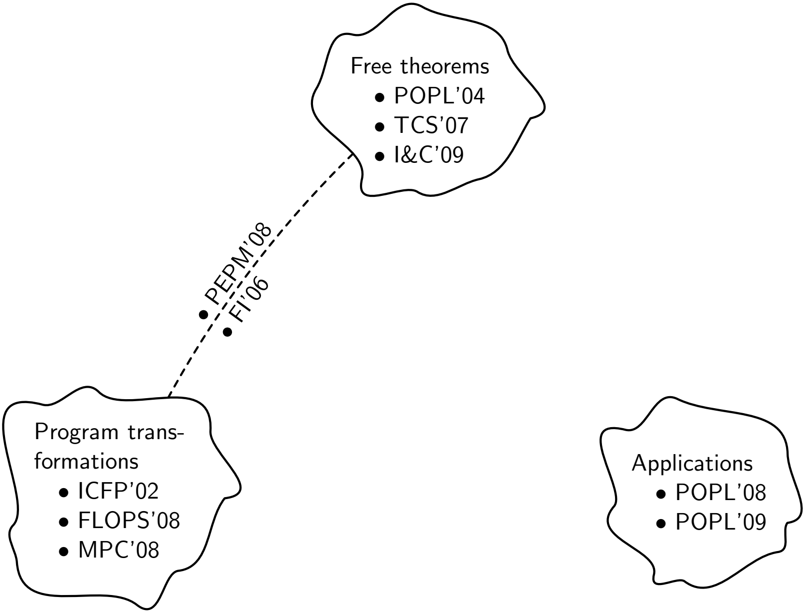

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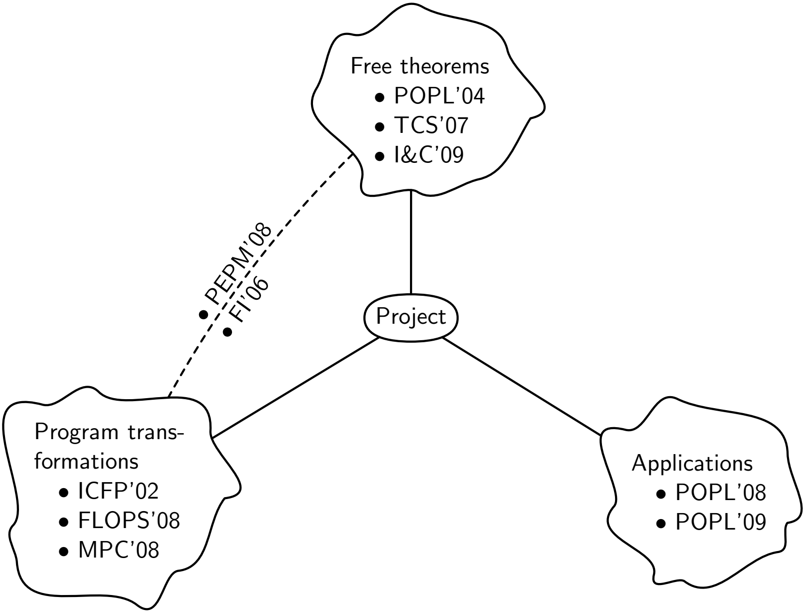

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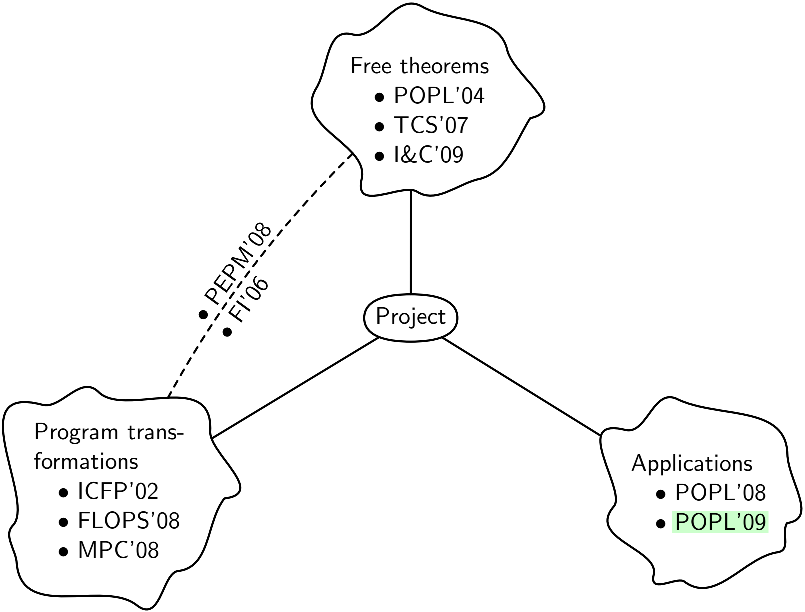
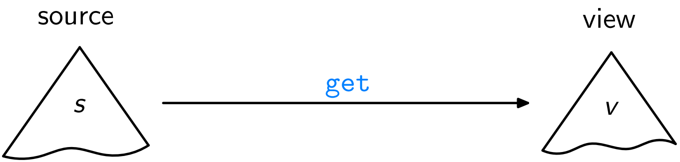


Bidirectional Transformation



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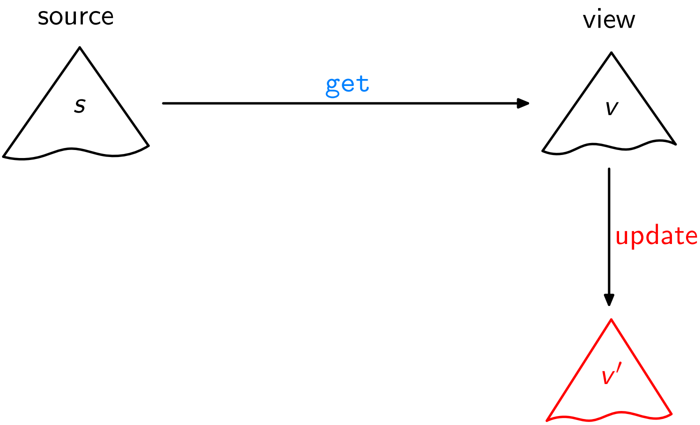

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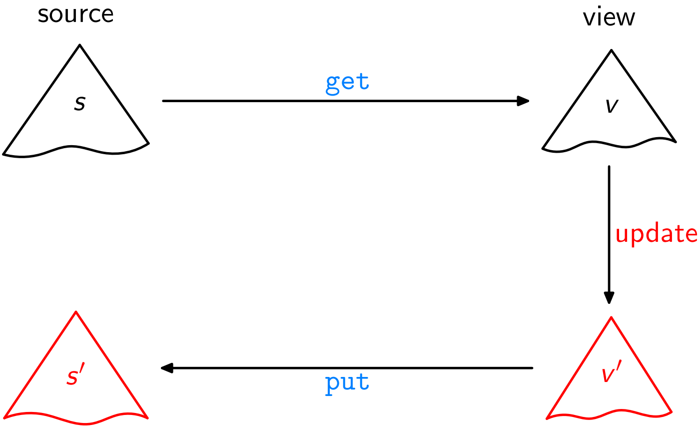

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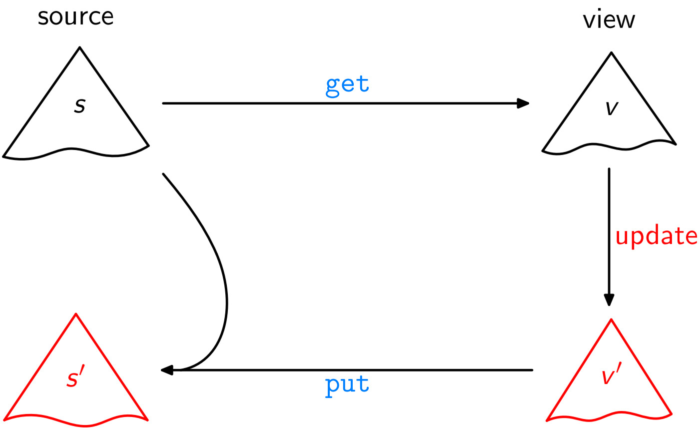

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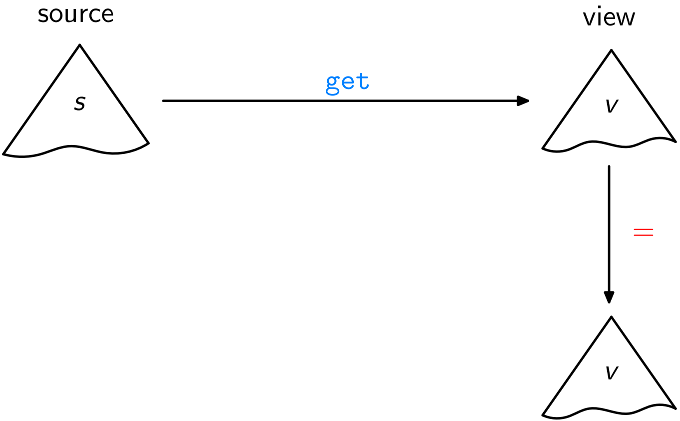
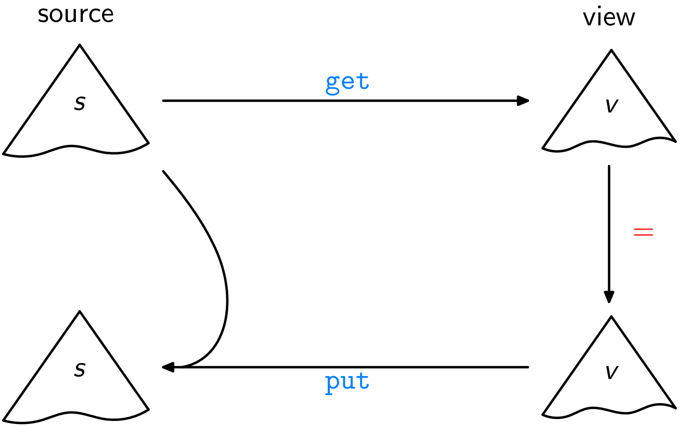


Acceptability / GetPut

Bidirectional Transformation



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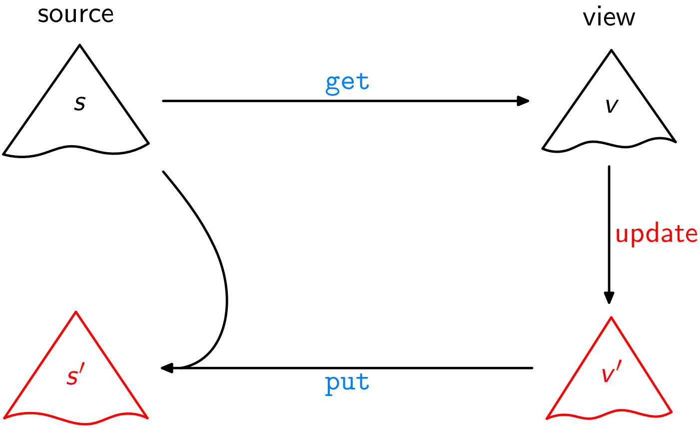

Consistency / PutGet

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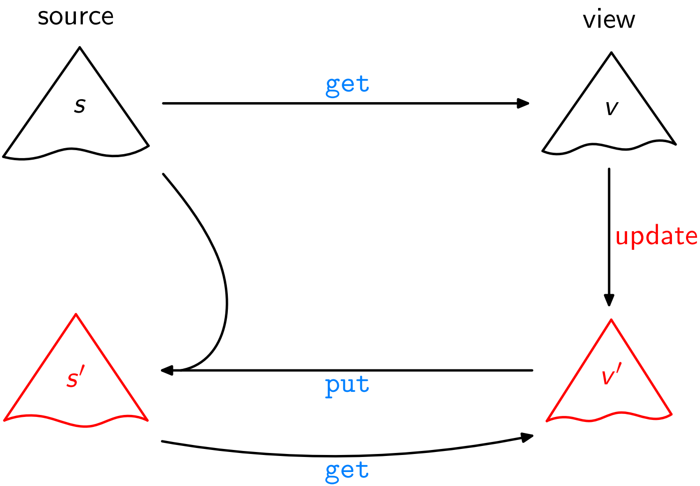

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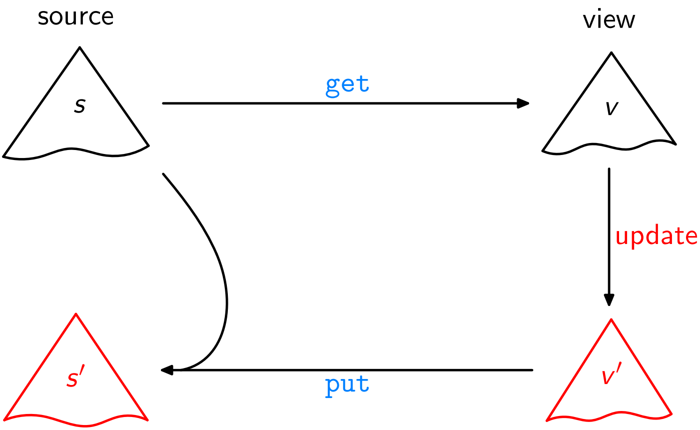

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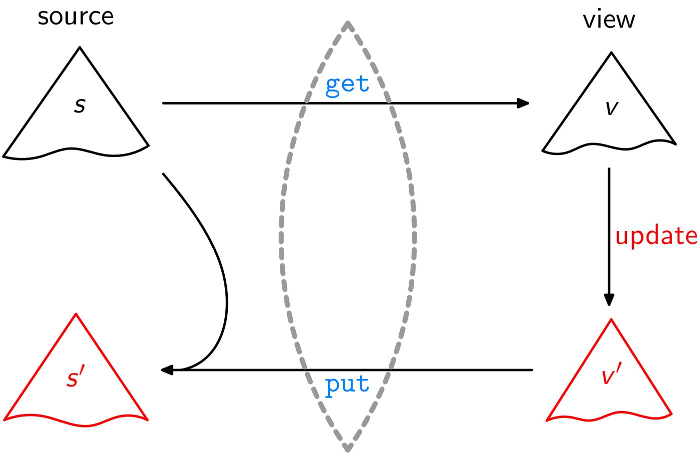
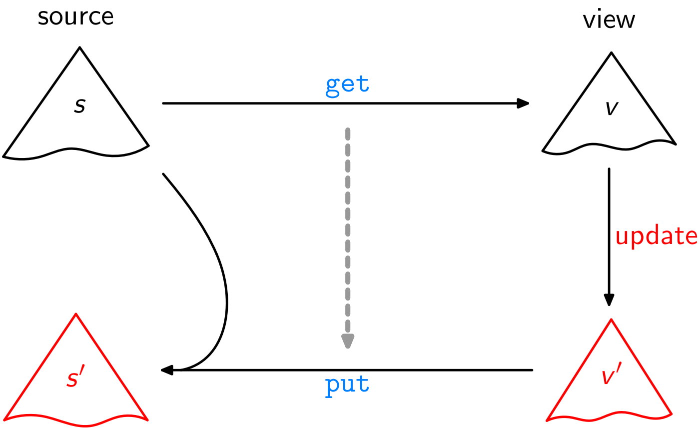


Lenses, DSLs

[Foster et al., ACM TOPLAS'07, ...]

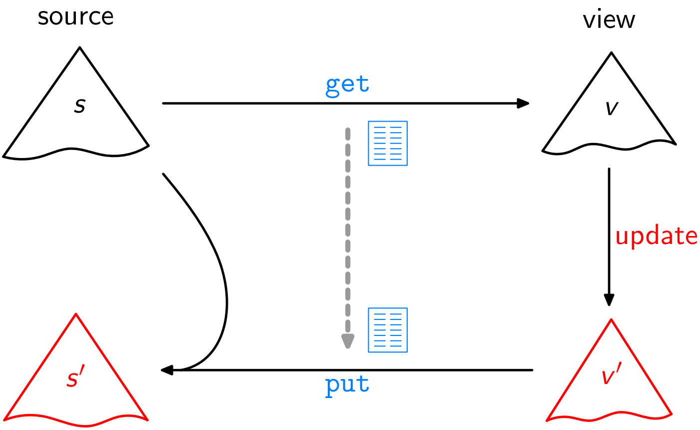
Bidirectional Transformation



Bidirectionalization

[Matsuda et al., ICFP'07]

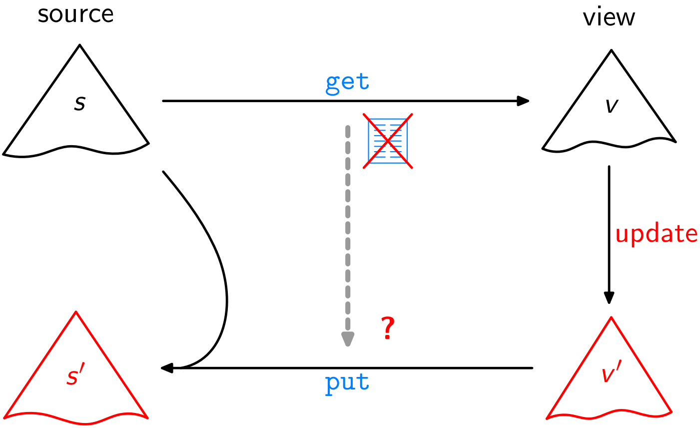
Bidirectional Transformation



Syntactic Bidirectionalization

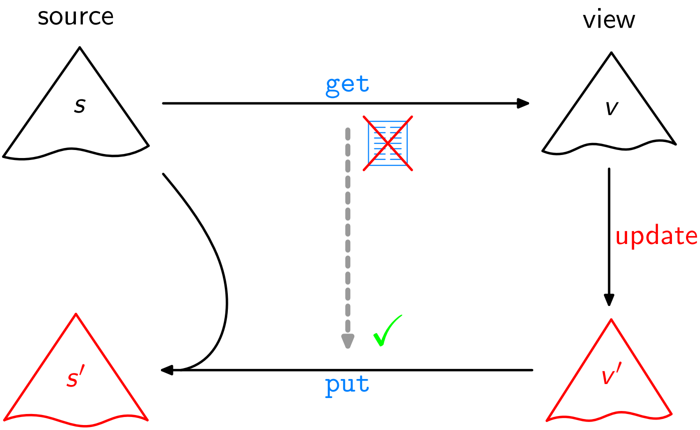
[Matsuda et al., ICFP'07]

Bidirectional Transformation



Semantic Bidirectionalization

Bidirectional Transformation



Semantic Bidirectionalization

[V., POPL'09]

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Aim: Write a higher-order function `bff` such that any `get` and `bff get` satisfy GetPut, PutGet, ...

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Examples:

$$\text{“abc”} \xrightarrow{\texttt{tail}} \text{“bc”}$$

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Examples:

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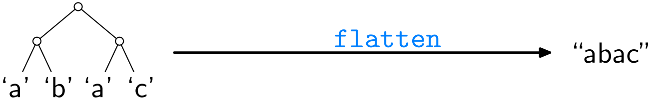

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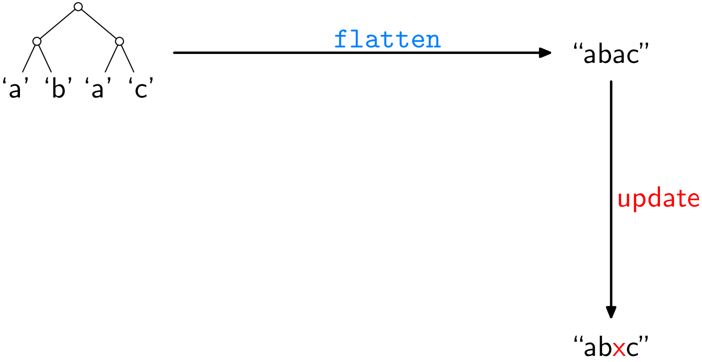

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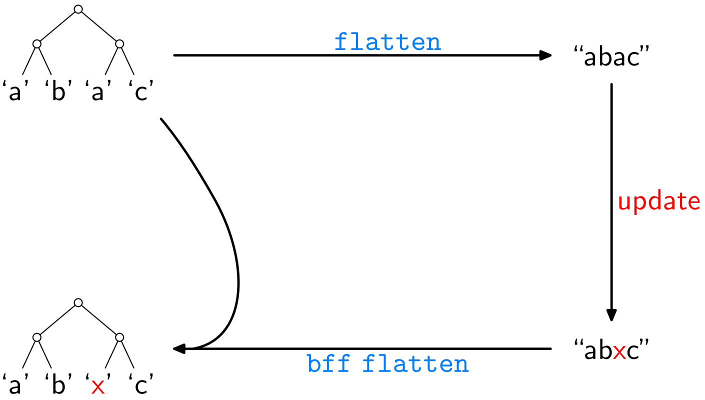

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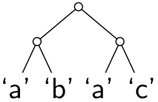

$\xrightarrow{\texttt{nub} \circ \texttt{flatten}}$ “abc”

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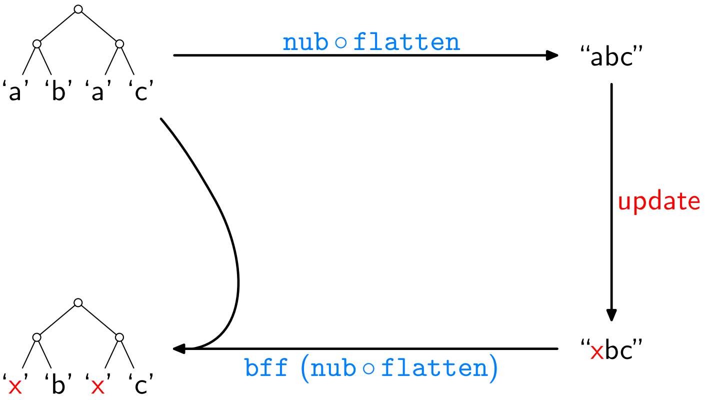

Analyzing Specific Instances

Assume we are given some

$$\texttt{get} :: [\alpha] \to [\alpha]$$

How can we, or `bff`, analyze it without access to its source code?

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$$\texttt{get} :: [\alpha] \rightarrow [\alpha]$$

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Idea: How about applying `get` to some input?

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$$\texttt{get} :: [\alpha] \to [\alpha]$$

How can we, or `bff`, analyze it without access to its source code?

Idea: How about applying `get` to some input?

Like:

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Analyzing Specific Instances

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Then transfer the gained insights to source lists other than $[0..n]$!

Using a Free Theorem

For every

$$\texttt{get} :: [\alpha] \to [\alpha]$$

we have

$$\texttt{map}\ f\ (\texttt{get}\ l) \quad = \quad \texttt{get}\ (\texttt{map}\ f\ l)$$

for arbitrary f and l, where

$$\begin{aligned}
&\texttt{map} :: (\alpha \to \beta) \to [\alpha] \to [\beta] \\
&\texttt{map}\ f\ [\,] \qquad\quad = [\,] \\
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Given an arbitrary list s of length $n+1$, set $f = (s\,!!)$, $l = [0..n]$, leading to:

$$\texttt{map}\ (s\,!!)\ (\texttt{get}\ [0..n]) \quad = \quad \texttt{get}\ (\texttt{map}\ (s\,!!)\ [0..n])$$

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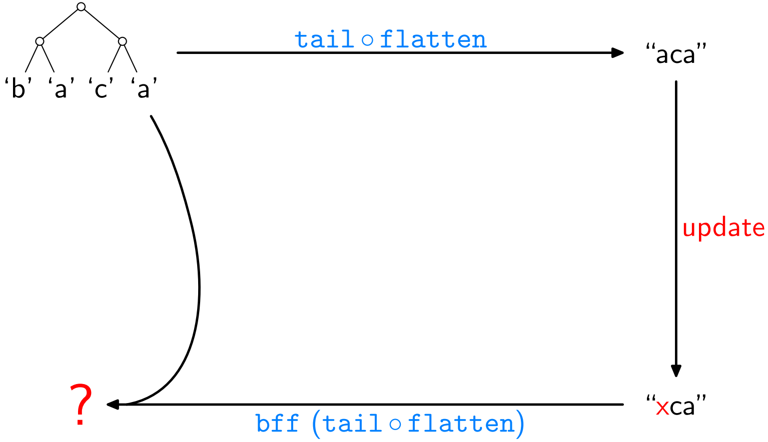
The Resulting Bidirectionalization Scheme by Example



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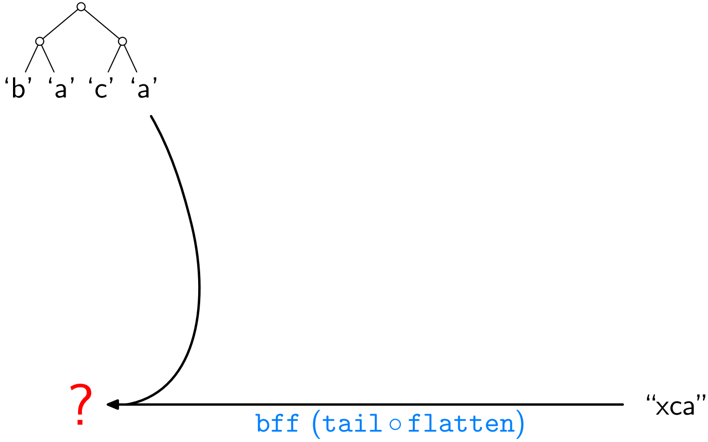

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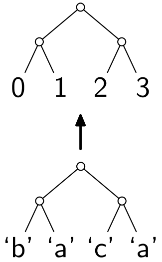

"xca"

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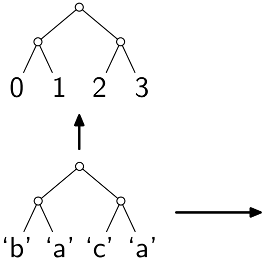

0	$\rightarrow$ 'b'
1	$\rightarrow$ 'a'
2	$\rightarrow$ 'c'
3	$\rightarrow$ 'a'

"xca"

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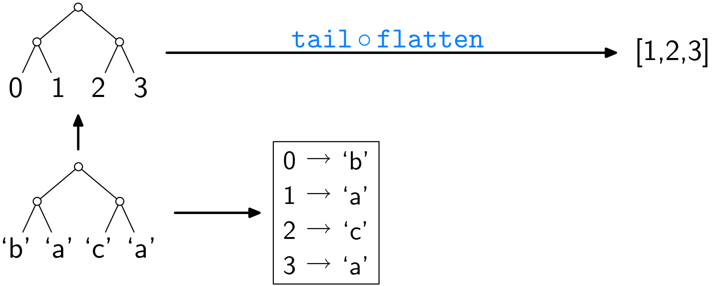

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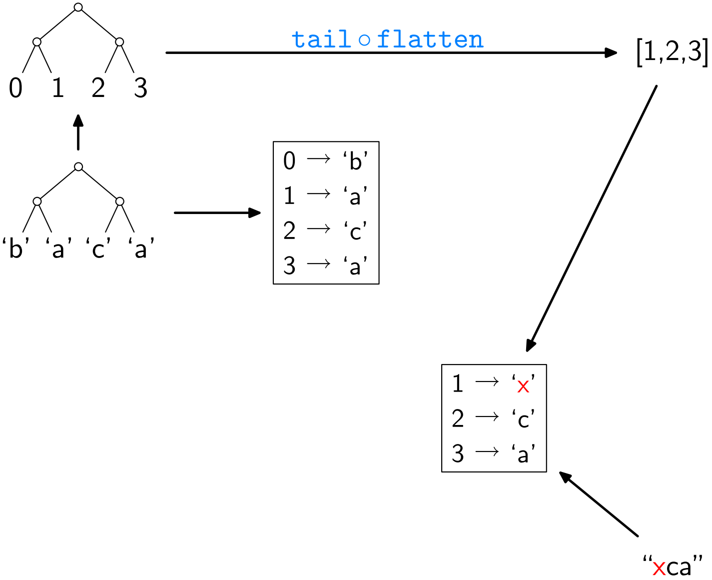

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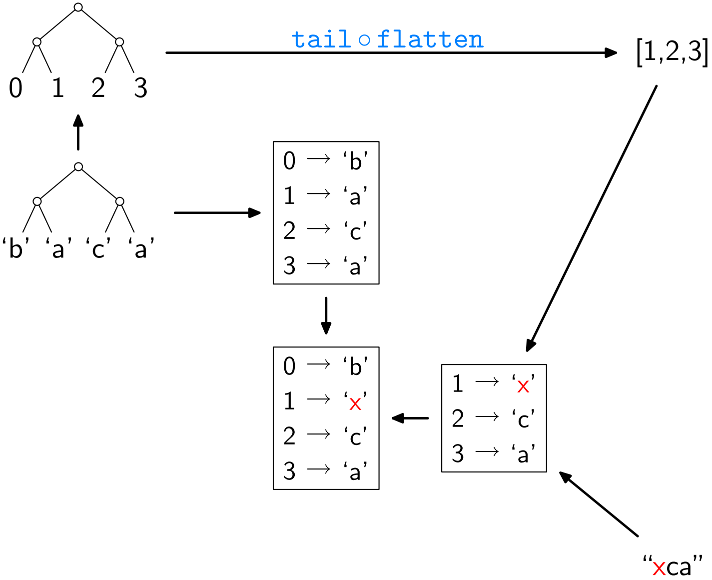

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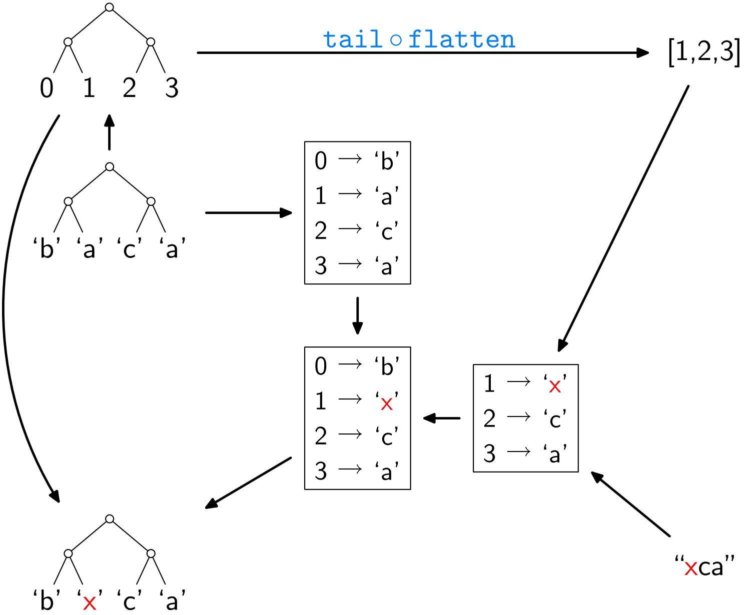

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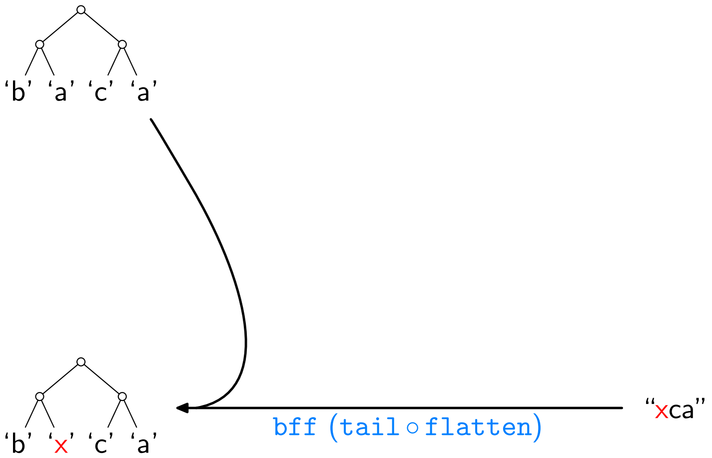

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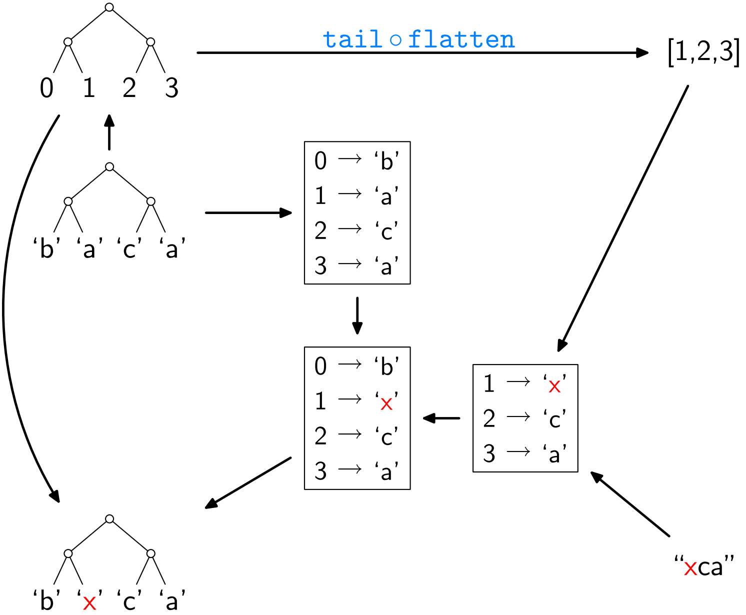

The Implementation (here: lists only, inefficient version)

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\end{aligned}
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&\qquad\qquad\qquad\qquad\qquad\quad\ \mathrm{Nothing}\qquad\quad\ \ \to (i, b) : m\\
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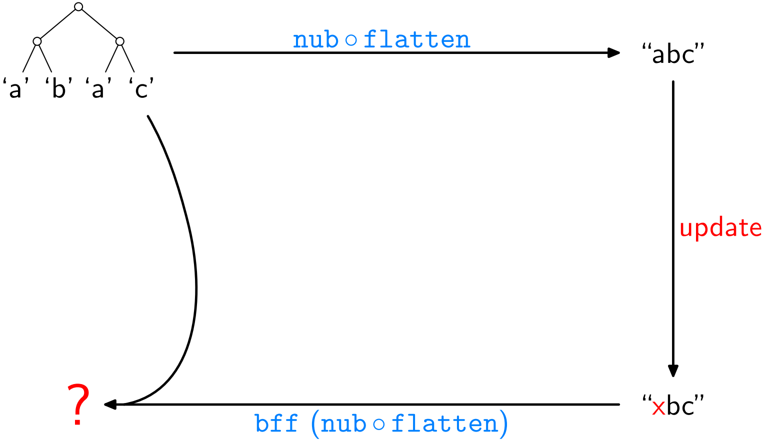
- actual code only slightly more elaborate
- online: `http://linux.tcs.inf.tu-dresden.de/~bff`

Another Interesting Example



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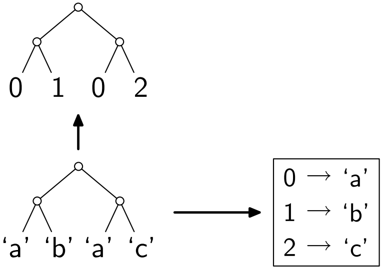

"xbc"

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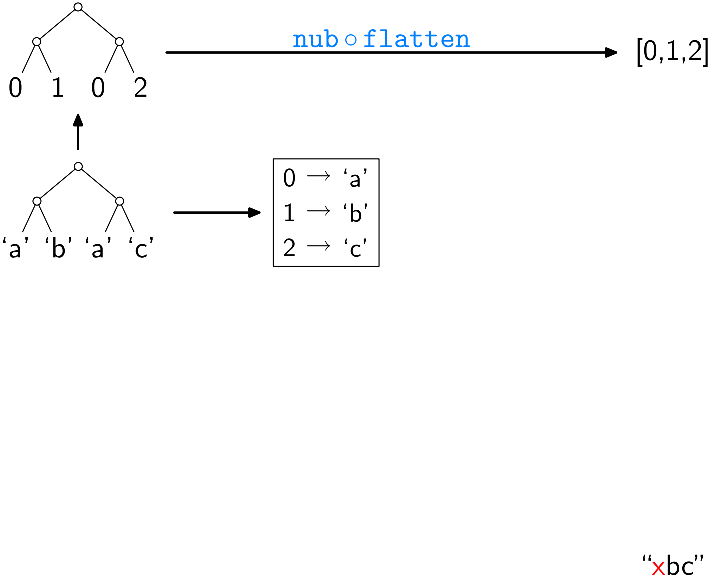

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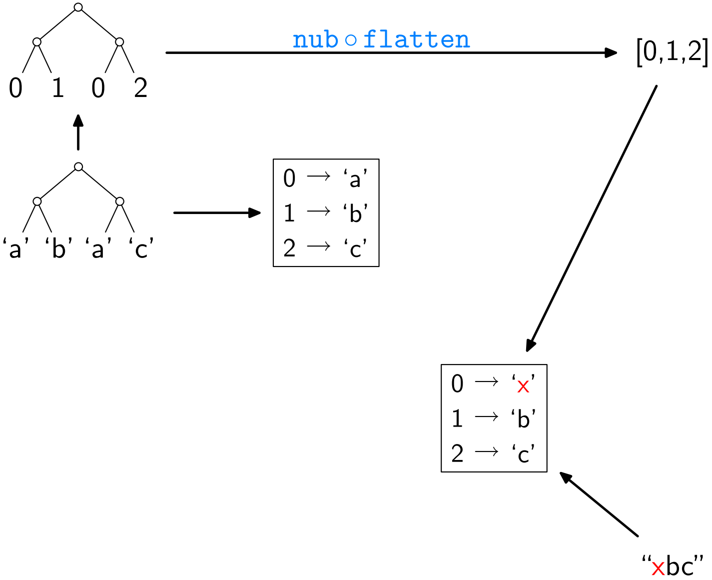

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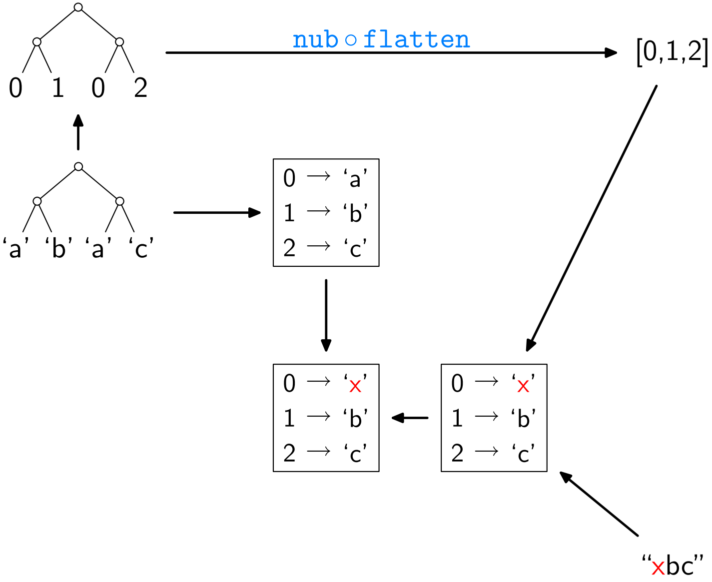

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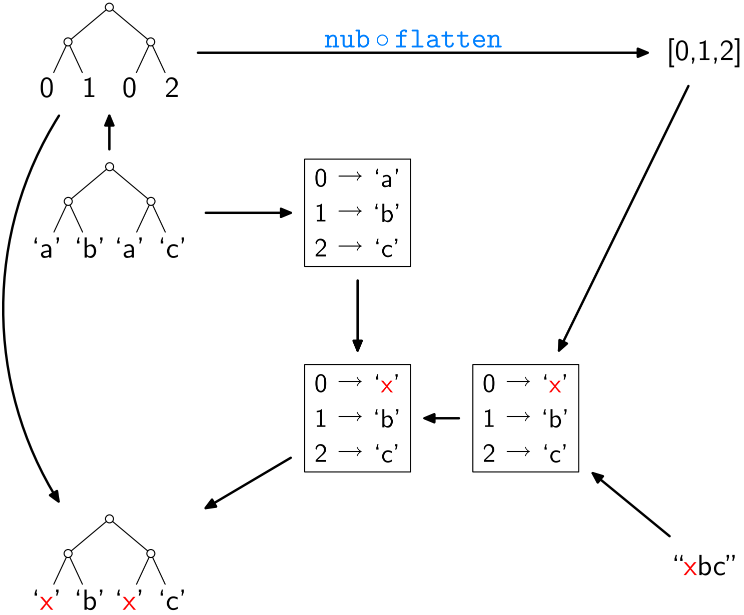

Summary and Outlook

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On the practical side:

- efficiency-improving program transformations
- applications in specific domains

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