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Concatenate, Reverse and Map Vanish For Free

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List-Producers using $+\!\!+$, *reverse* and *map*:

$$part\ even\ [1..10] = [2,4,6,8,10,1,3,5,7,9]$$

$$
\begin{array}{l}
part :: (\alpha \rightarrow \mathsf{Bool}) \rightarrow [\alpha] \rightarrow [\alpha] \\
part\ p\ l = \text{let}\ f\ []\ z = z \\
\qquad\qquad\quad f\ (x : xs)\ z = \text{if}\ p\ x\ \text{then}\ x : (f\ xs\ z) \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{else}\ f\ xs\ (z +\!\!+ [x]) \\
\qquad\qquad \text{in}\ f\ l\ []
\end{array}
$$

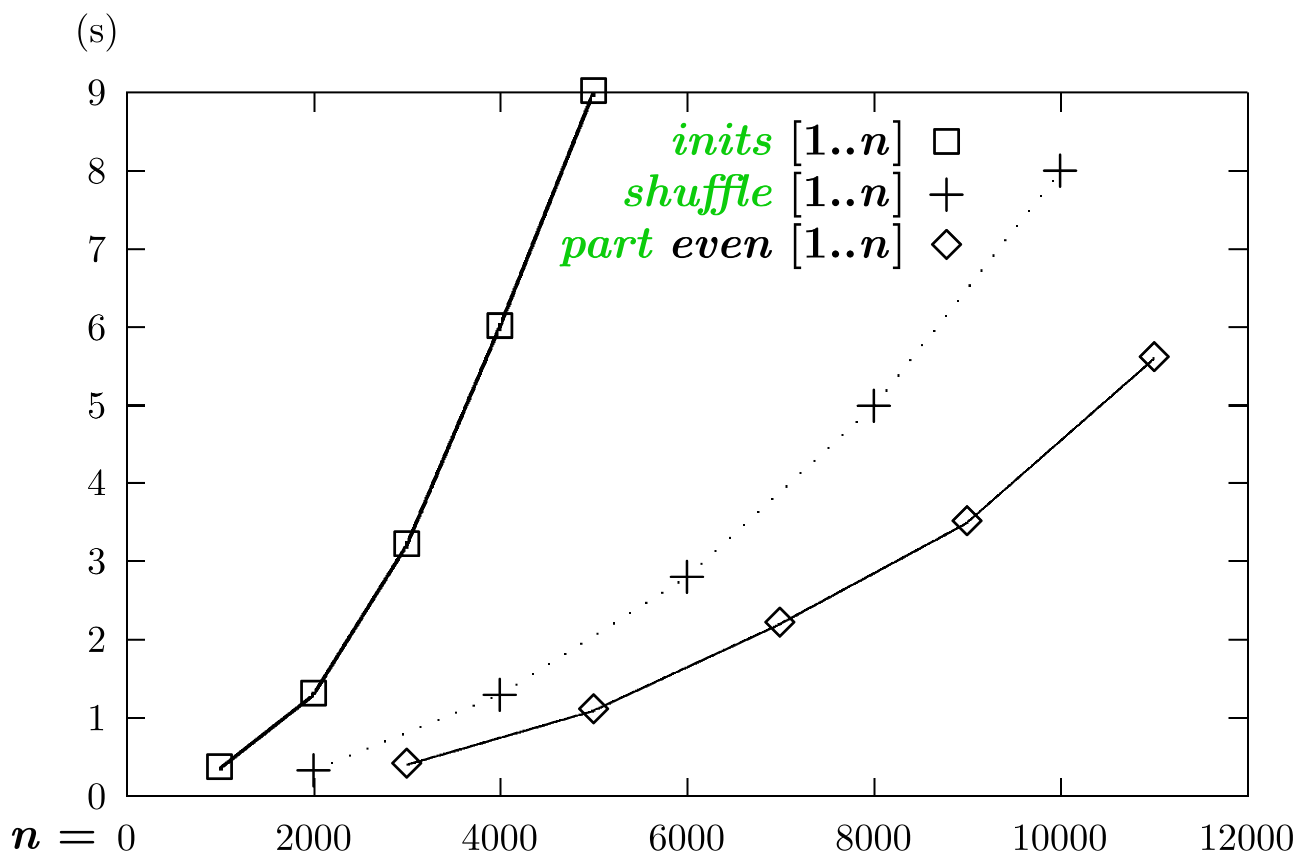
$$shuffle\ \text{“whatever”} = \text{“waeervth”}$$

$$
\begin{array}{l}
shuffle :: [\alpha] \rightarrow [\alpha] \\
shuffle\ [] = [] \\
shuffle\ (x : xs) = x : (reverse\ (shuffle\ xs))
\end{array}
$$

$$inits\ [1..4] = [[],[1],[1,2],[1,2,3],[1,2,3,4]]$$

$$
\begin{array}{l}
inits :: [\alpha] \rightarrow [[\alpha]] \\
inits\ [] = [[]] \\
inits\ (x : xs) = [] : (map\ (x :)\ (inits\ xs))
\end{array}
$$

Runtimes dominated by repeated List-Operations:



Efficiency by List Abstraction (*part*):

$$
\begin{aligned}
&\mathit{part} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha] \\
&\mathit{part}\ p\ l = \text{let } f\ [\,]\ z = z \\
&\qquad\qquad\qquad f\ (x : xs)\ z = \text{if } p\ x \text{ then } x : (f\ xs\ z) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{else } f\ xs\ (z +\!\!+ (x : [\,])) \\
&\qquad\qquad\quad \text{in } f\ l\ [\,]
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\mathit{part}^{\star} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha] \\
&\mathit{part}^{\star}\ p\ l = \mathit{vanish}_{+\!+}\ (\lambda n\ c\ a \to \\
&\qquad\qquad\quad \text{let } f\ [\,]\ z = z \\
&\qquad\qquad\qquad f\ (x : xs)\ z = \text{if } p\ x \text{ then } x\ \text{`}c\text{`}\ (f\ xs\ z) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{else } f\ xs\ (z\ \text{`}a\text{`}\ (x\ \text{`}c\text{`}\ n)) \\
&\qquad\qquad\quad \text{in } f\ l\ n)
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{vanish}_{+\!+} :: (\forall \beta\,.\,\beta \to (\alpha \to \beta \to \beta) \\
&\qquad\qquad\qquad \to (\beta \to \beta \to \beta) \to \beta) \to [\alpha] \\
&\mathit{vanish}_{+\!+}\ g = g\ [\,]\ (:)\ (+\!\!+)
\end{aligned}
$$

Such list abstraction can be performed *automatically*, based on the rank-2 polymorphic type of $\mathit{vanish}_{+\!+}$ and partial type inference [Chitil, 1999]!

Runtimes: $n =$	3000	5000	7000	9000	11000
part *even* $[1..n]$	0.4	1.1	2.2	3.5	5.6 (s)
$\mathit{part}^{\star}$ *even* $[1..n]$	0.004	0.006	0.009	0.012	0.015 (s)

Efficiency by List Abstraction (*shuffle*):

$$
\begin{aligned}
&\mathit{shuffle} :: [\alpha] \longrightarrow [\alpha] \\
&\mathit{shuffle}\ []\ = []\\
&\mathit{shuffle}\ (x : xs) = x : (\mathit{reverse}\ (\mathit{shuffle}\ xs))
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\mathit{shuffle}^\star :: [\alpha] \longrightarrow [\alpha] \\
&\mathit{shuffle}^\star\ l = \mathit{vanish}_{rev}\ (\lambda n\ c\ r \longrightarrow \\
&\qquad \text{let } f\ []\ = n \\
&\qquad\quad f\ (x : xs) = x\ `c`\ (r\ (f\ xs)) \\
&\qquad \text{in } f\ l)
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{vanish}_{rev} :: (\forall \beta\,.\,\beta \longrightarrow (\alpha \longrightarrow \beta \longrightarrow \beta) \longrightarrow (\beta \longrightarrow \beta) \longrightarrow \beta) \longrightarrow [\alpha] \\
&\mathit{vanish}_{rev}\ g \sqsupseteq g\ []\ (:)\ \mathit{reverse}
\end{aligned}
$$

Runtimes: $n =$	2000	4000	6000	8000	10000
shuffle $[1..n]$	0.33	1.3	2.8	5.0	8.0 (s)
shuffle* $[1..n]$	0.005	0.01	0.016	0.02	0.025 (s)

Efficiency by List Abstraction (*inits*):

$$\begin{aligned}
&inits :: [\alpha] \to [[\alpha]]\\
&inits\ [] = [] : []\\
&inits\ (x : xs) = [] : (map\ (x :)\ (inits\ xs))
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
&inits^{\star} :: [\alpha] \to [[\alpha]]\\
&inits^{\star}\ l = vanish_{++,rev,map}\ (\lambda n\ c\ a\ r\ m \to\\
&\qquad \text{let } f\ [] = []\ `c`\ n\\
&\qquad\qquad f\ (x : xs) = []\ `c`\ (m\ (x :)\ (f\ xs))\\
&\qquad \text{in } f\ l)
\end{aligned}$$

$$\begin{aligned}
&vanish_{++,rev,map} :: (\forall\beta\ .\ \cdots) \to [\alpha]\\
&vanish_{++,rev,map}\ g \sqsupseteq g\ []\ (:)\ (++)\ reverse\ map
\end{aligned}$$

Runtimes: $n =$	1000	2000	3000	4000	5000
inits $[1..n]$	0.35	1.3	3.2	6.0	9.0 (s)
inits* $[1..n]$	0.08	0.3	0.7	1.3	2.0 (s)

Actual Definitions of the *vanish*-Combinators:

$$vanish_{++} :: (\forall \beta . \beta \to (\alpha \to \beta \to \beta) \to (\beta \to \beta \to \beta) \to \beta) \to [\alpha]$$
$$vanish_{++}\ g \ = \ g\ id\ (\lambda x\ h\ ys \to x : (h\ ys))\ (\circ)\ []$$

$$vanish_{rev} :: (\forall \beta . \beta \to (\alpha \to \beta \to \beta) \to (\beta \to \beta) \to \beta) \to [\alpha]$$
$$\begin{aligned} vanish_{rev}\ g \ = \ & fst\ (g\ (\lambda ys \to (ys, ys)) \\ & \quad (\lambda x\ h\ ys \to (x : (fst\ (h\ ys)), snd\ (h\ (x : ys)))) \\ & \quad (\lambda h\ ys \to swap\ (h\ ys))\ []) \end{aligned}$$

$$\begin{aligned} vanish_{++,rev,map} :: \ & (\forall \beta . \beta \to (\alpha \to \beta \to \beta) \to (\beta \to \beta \to \beta) \to (\beta \to \beta) \\ & \to ((\alpha \to \alpha) \to \beta \to \beta) \to \beta) \to [\alpha] \end{aligned}$$
$$\begin{aligned} vanish_{++,rev,map}\ g \ = \ & fst\ (g\ (\lambda f\ ys \to (ys, ys)) \\ & \quad (\lambda x\ h\ f\ ys \to ((f\ x) : (fst\ (h\ f\ ys)), \\ & \qquad\qquad snd\ (h\ f\ ((f\ x) : ys)))) \\ & \quad (\lambda h_1\ h_2\ f\ ys \to (fst\ (h_1\ f\ (fst\ (h_2\ f\ ys))), \\ & \qquad\qquad snd\ (h_2\ f\ (snd\ (h_1\ f\ ys))))) \\ & \quad (\lambda h\ f\ ys \to swap\ (h\ f\ ys)) \\ & \quad (\lambda k\ h\ f\ ys \to h\ (f \circ k)\ ys)\ id\ []) \end{aligned}$$

User-Exposed Semantics of the *vanish*-Combinators:

$$vanish_{++} :: (\forall \beta \,.\, \beta \rightarrow (\alpha \rightarrow \beta \rightarrow \beta) \rightarrow (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta) \rightarrow [\alpha]$$
$$vanish_{++}\ g \ = \ g\ []\ (:)\ (++)$$

$$vanish_{rev} :: (\forall \beta \,.\, \beta \rightarrow (\alpha \rightarrow \beta \rightarrow \beta) \rightarrow (\beta \rightarrow \beta) \rightarrow \beta) \rightarrow [\alpha]$$
$$vanish_{rev}\ g \ \sqsupseteq \ g\ []\ (:)\ reverse$$

$$vanish_{++,rev,map} :: (\forall \beta \,.\, \beta \rightarrow (\alpha \rightarrow \beta \rightarrow \beta) \rightarrow (\beta \rightarrow \beta \rightarrow \beta) \rightarrow (\beta \rightarrow \beta) \rightarrow ((\alpha \rightarrow \alpha) \rightarrow \beta \rightarrow \beta) \rightarrow \beta) \rightarrow [\alpha]$$
$$vanish_{++,rev,map}\ g \ \sqsupseteq \ g\ []\ (:)\ (++)\ reverse\ map$$

Proven using *free theorems* [Wadler, 1989], driven by the algebraic laws:

$$(xs \mathbin{++} ys) \mathbin{++} zs = xs \mathbin{++} (ys \mathbin{++} zs) \qquad (1)$$
$$reverse\ (reverse\ xs) \sqsubseteq xs \qquad (2)$$
$$map\ f\ (map\ k\ xs) = map\ (f \circ k)\ xs \qquad (3)$$

Proof: $vanish_{++}\ g = g\ []\ (:)\ (++)$

Parametricity [Reynolds, 1983] gives for the type of

$$g :: \forall \beta\ .\ \beta \rightarrow (\mathsf{A} \rightarrow \beta \rightarrow \beta) \rightarrow (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$$

the following *free theorem* [Wadler, 1989]:

$$\begin{aligned} & (n, n') \in \mathcal{R} \ \wedge\ (\forall x :: \mathsf{A}, (l, l') \in \mathcal{R}\ .\ (c\ x\ l, c'\ x\ l') \in \mathcal{R}) \\ \wedge\ & (\forall (l_1, l_1') \in \mathcal{R}, (l_2, l_2') \in \mathcal{R}\ .\ (a\ l_1\ l_2, a'\ l_1'\ l_2') \in \mathcal{R}) \\ \Rightarrow\ & (g\ n\ c\ a, g\ n'\ c'\ a') \in \mathcal{R}. \end{aligned}$$

Instantiate with $n = []$, $c = (:)$, $a = (++)$, $n' = id$, $c' = (\lambda x\ h\ ys \rightarrow x : (h\ ys))$, $a' = (\circ)$, and $\mathcal{R} = \{(l, l') \mid \forall ys :: [\mathsf{A}]\ .\ l ++ ys = l'\ ys\}$:

$$\begin{aligned} & (\forall ys\ .\ [] ++ ys = ys) \\ \wedge\ & (\forall x, l, l'\ .\ (\forall ys\ .\ l ++ ys = l'\ ys) \ \Rightarrow\ (\forall ys\ .\ (x : l) ++ ys = x : (l'\ ys))) \\ \wedge\ & (\forall l_1, l_1', l_2, l_2'\ .\ (\forall ys\ .\ l_1 ++ ys = l_1'\ ys) \ \wedge\ (\forall ys\ .\ l_2 ++ ys = l_2'\ ys) \\ & \quad \Rightarrow\ (\forall ys\ .\ (l_1 ++ l_2) ++ ys = l_1'\ (l_2'\ ys))) \\ \Rightarrow\ & (\forall ys\ .\ (g\ []\ (:)\ (++)) ++ ys = g\ id\ (\lambda x\ h\ ys \rightarrow x : (h\ ys))\ (\circ)\ ys). \end{aligned}$$

The preconditions of this implication are fulfilled by the definition of $(++)$ and by law (1), hence: $(g\ []\ (:)\ (++)) ++ [] = vanish_{++}\ g$.

A general Methodology (e.g.: the *filter* vanishes)

$$\begin{aligned}
&nub :: \mathsf{Eq}\ \alpha \Rightarrow [\alpha] \rightarrow [\alpha] \\
&nub\ [] = [] \\
&nub\ (x : xs) = x : (filter\ (x \neq)\ (nub\ xs))
\end{aligned}$$

1. Freezing and Efficient Conversion:

$$\begin{aligned}
&\text{data } \mathsf{List}\ \alpha = \mathsf{Nil} \mid \mathsf{Cons}\ \alpha\ (\mathsf{List}\ \alpha) \mid \mathsf{Filter}\ (\alpha \rightarrow \mathsf{Bool})\ (\mathsf{List}\ \alpha) \\
&nub' :: \mathsf{Eq}\ \alpha \Rightarrow [\alpha] \rightarrow \mathsf{List}\ \alpha \\
&nub'\ [] = \mathsf{Nil} \\
&nub'\ (x : xs) = \mathsf{Cons}\ x\ (\mathsf{Filter}\ (x \neq)\ (nub'\ xs))
\end{aligned}$$

$$\begin{aligned}
&convert^\star :: \mathsf{List}\ \alpha \rightarrow [\alpha] \\
&convert^\star\ l = \text{let } h\ \mathsf{Nil}\ p = [] \\
&\qquad h\ (\mathsf{Cons}\ x\ xs)\ p = \text{if } (p\ x) \text{ then } (x : (h\ xs\ p)) \text{ else } (h\ xs\ p) \\
&\qquad h\ (\mathsf{Filter}\ q\ xs)\ p = h\ xs\ (\lambda x \rightarrow q\ x\ \&\&\ p\ x) \\
&\quad \text{in } h\ l\ (\lambda x \rightarrow \mathsf{True})
\end{aligned}$$

2. Preparing Shortcut Fusion [Gill *et al.*, 1993]:

$$build_{\mathsf{List}}\ g \ = \ g\ \mathsf{Nil}\ \mathsf{Cons}\ \mathsf{Filter}$$

$$\begin{aligned}
&nub' :: \mathsf{Eq}\ \alpha \Rightarrow [\alpha] \to \mathsf{List}\ \alpha\\
&nub'\ l = build_{\mathsf{List}}\ (\lambda n\ c\ f \to \text{let } h\ [\,] = n\\
&\qquad\qquad\qquad\qquad\qquad\quad h\ (x : xs) = c\ x\ (f\ (x \neq)\ (h\ xs))\\
&\qquad\qquad\qquad\qquad\quad \text{in } h\ l)
\end{aligned}$$

$$\begin{aligned}
fold_{\mathsf{List}}\ \mathsf{Nil}\ n\ c\ f &= n\\
fold_{\mathsf{List}}\ (\mathsf{Cons}\ x\ xs)\ n\ c\ f &= c\ x\ (fold_{\mathsf{List}}\ xs\ n\ c\ f)\\
fold_{\mathsf{List}}\ (\mathsf{Filter}\ q\ xs)\ n\ c\ f &= f\ q\ (fold_{\mathsf{List}}\ xs\ n\ c\ f)
\end{aligned}$$

$$\begin{aligned}
&convert^{\star} :: \mathsf{List}\ \alpha \to [\alpha]\\
&convert^{\star}\ l = fold_{\mathsf{List}}\ l\\
&\qquad\qquad (\lambda p \to [\,])\\
&\qquad\qquad (\lambda x\ h\ p \to \text{if } (p\ x) \text{ then } (x : (h\ p)) \text{ else } (h\ p))\\
&\qquad\qquad (\lambda q\ h\ p \to h\ (\lambda x \to q\ x\ \&\&\ p\ x))\\
&\qquad\qquad (\lambda x \to \mathsf{True})
\end{aligned}$$

3. Calculate using Fusion Law: $fold_{\mathsf{List}}\ (build_{\mathsf{List}}\ g) = g$

$$
\begin{aligned}
& convert^{\star}\ (nub'\ l) \\
= {} & fold_{\mathsf{List}}\ (build_{\mathsf{List}}\ (\lambda n\ c\ f \rightarrow \text{let } h\ [\,] = n \\
& \qquad\qquad\qquad\qquad\qquad\quad h\ (x : xs) = c\ x\ (f\ (x \neq)\ (h\ xs)) \\
& \qquad\qquad\qquad\qquad\qquad \text{in } h\ l)) \\
& \qquad (\lambda p \rightarrow [\,]) \\
& \qquad (\lambda x\ h\ p \rightarrow \text{if } (p\ x) \text{ then } (x : (h\ p)) \text{ else } (h\ p)) \\
& \qquad (\lambda q\ h\ p \rightarrow h\ (\lambda x \rightarrow q\ x\ \&\&\ p\ x)) \\
& \qquad (\lambda x \rightarrow \mathsf{True}) \\
= {} & (\lambda n\ c\ f \rightarrow \text{let } h\ [\,] = n \\
& \qquad\qquad\quad h\ (x : xs) = c\ x\ (f\ (x \neq)\ (h\ xs)) \\
& \qquad\qquad \text{in } h\ l) \\
& (\lambda p \rightarrow [\,]) \\
& (\lambda x\ h\ p \rightarrow \text{if } (p\ x) \text{ then } (x : (h\ p)) \text{ else } (h\ p)) \\
& (\lambda q\ h\ p \rightarrow h\ (\lambda x \rightarrow q\ x\ \&\&\ p\ x)) \\
& (\lambda x \rightarrow \mathsf{True})
\end{aligned}
$$

4. Abstract into Combinator:

$$vanish_{\mathit{filter}}\ g \;=\; g\ (\lambda p \rightarrow [\,])\ (\lambda x\ h\ p \rightarrow \text{if } (p\ x) \text{ then } (x : (h\ p)) \text{ else } (h\ p))$$
$$(\lambda q\ h\ p \rightarrow h\ (\lambda x \rightarrow q\ x\ \&\&\ p\ x))\ (\lambda x \rightarrow \mathsf{True})$$

$$nub^{\star} :: \mathsf{Eq}\ \alpha \;\Rightarrow\; [\alpha] \;\rightarrow\; [\alpha]$$
$$\begin{aligned} nub^{\star}\ l \;=\; vanish_{\mathit{filter}}\ (\lambda n\ c\ f \rightarrow \text{let } & h\ [\,] = n \\ & h\ (x : xs) = c\ x\ (f\ (x \neq)\ (h\ xs)) \\ \text{in } & h\ l) \end{aligned}$$

5. Prove Correctness:

$$vanish_{\mathit{filter}} :: (\forall \beta . \beta \rightarrow (\alpha \rightarrow \beta \rightarrow \beta) \rightarrow ((\alpha \rightarrow \mathsf{Bool}) \rightarrow \beta \rightarrow \beta) \rightarrow \beta) \rightarrow [\alpha]$$
$$vanish_{\mathit{filter}}\ g \;=\; g\ [\,]\ (:)\ \mathit{filter}$$

Using a *free theorem* and the following law:

$$\mathit{filter}\ p\ (\mathit{filter}\ q\ xs) = \mathit{filter}\ (\lambda x \rightarrow q\ x\ \&\&\ p\ x)\ xs \qquad (4)$$

Proof: $\mathit{vanish}_{\mathit{filter}}\ g = g\ []\ (:)\ \mathit{filter}$

From the type of g follows the *free theorem*:

$$
\begin{array}{ll}
 & (n, n') \in \mathcal{R} \\
\wedge & (\forall x :: \mathsf{A}, (l, l') \in \mathcal{R}\ .\ (c\ x\ l, c'\ x\ l') \in \mathcal{R}) \\
\wedge & (\forall q :: \mathsf{A} \to \mathsf{Bool}, (l, l') \in \mathcal{R}\ .\ (f\ q\ l, f'\ q\ l') \in \mathcal{R}) \\
\Rightarrow & (g\ n\ c\ f, g\ n'\ c'\ f') \in \mathcal{R}.
\end{array}
$$

Instantiate with $n = []$, $c = (:)$, $f = \mathit{filter}$, $n' = (\lambda p \to [])$, $c' = (\lambda x\ h\ p \to \text{if } (p\ x) \text{ then } (x : (h\ p)) \text{ else } (h\ p))$, $f' = (\lambda q\ h\ p \to h\ (\lambda x \to q\ x\ \&\&\ p\ x))$, and $\mathcal{R} = \{(l, l') \mid \forall p :: \mathsf{A} \to \mathsf{Bool}\ .\ \mathit{filter}\ p\ l = l'\ p\}$:

$$
\begin{array}{ll}
 & (\forall p\ .\ \mathit{filter}\ p\ [] = []) \\
\wedge & (\forall x, l, l'\ .\ (\forall p\ .\ \mathit{filter}\ p\ l = l'\ p) \\
 & \quad \Rightarrow (\forall p\ .\ \mathit{filter}\ p\ (x : l) = \text{if } (p\ x) \text{ then } (x : (l'\ p)) \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{else } (l'\ p))) \\
\wedge & (\forall q, l, l'\ .\ (\forall p\ .\ \mathit{filter}\ p\ l = l'\ p) \\
 & \quad \Rightarrow (\forall p\ .\ \mathit{filter}\ p\ (\mathit{filter}\ q\ l) = l'\ (\lambda x \to q\ x\ \&\&\ p\ x))) \\
\Rightarrow & (\forall p\ .\ \mathit{filter}\ p\ (g\ []\ (:)\ \mathit{filter}) \\
 & \qquad = g\ (\lambda p \to []) \\
 & \qquad\qquad (\lambda x\ h\ p \to \text{if } (p\ x) \text{ then } (x : (h\ p)) \text{ else } (h\ p)) \\
 & \qquad\qquad (\lambda q\ h\ p \to h\ (\lambda x \to q\ x\ \&\&\ p\ x))\ p).
\end{array}
$$

The preconditions of this implication are fulfilled by the definition of *filter* and by law (4), hence:

$$\mathit{filter}\ (\lambda x \to \mathsf{True})\ (g\ []\ (:)\ \mathit{filter}) = \mathit{vanish}_{\mathit{filter}}\ g.$$

Summary:

- **Variation of list abstraction: abstract not only over data constructors, but also over manipulating operations.**
- **Methodology: "freezing" plus "efficient conversion as a *fold*" for synthesizing optimized list implementations.**
  (also applicable to other algebraic data types)
- **Encapsulate essence of optimizations in reusable rank-2 polymorphic combinators.**

$\Rightarrow$ **Allows automation and proofs using free theorems.**

References

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