Type-Based Reasoning and Imprecise Errors

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Polymorphic Types: An Example in Haskell

A standard function:

$$\begin{aligned}\texttt{map}\ f\ [\,] &= [\,]\\ \texttt{map}\ f\ (a : as) &= (f\ \ a) : (\texttt{map}\ f\ \ as)\end{aligned}$$

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$\texttt{map not}\ [1, 2, 3]$ $\lightning$ rejected at compile-time

Another Example

$$
\begin{array}{l}
\texttt{takeWhile} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha] \\
\texttt{takeWhile}\ p\ [\,] = [\,] \\
\texttt{takeWhile}\ p\ (a : as) \mid p\ a = a : (\texttt{takeWhile}\ p\ as) \\
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For every choice of p, f, and l:

$$\texttt{takeWhile}\ p\ (\texttt{map}\ f\ \ l) \ = \ \texttt{map}\ f\ (\texttt{takeWhile}\ (p \circ f)\ l)$$

Provable by induction.

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Or as a "free theorem" [Wadler, FPCA'89].

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- $g :: (\alpha \to \text{Bool}) \to [\alpha] \to [\alpha]$ must work uniformly for every instantiation of α.

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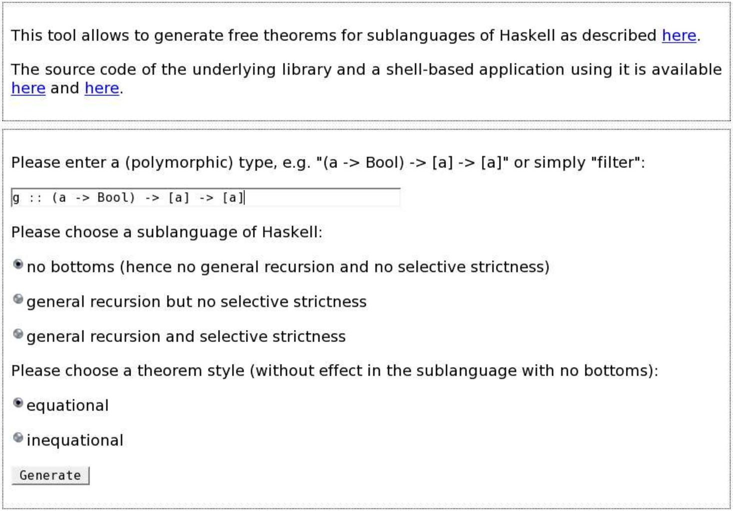
- That is what was claimed!

Automatic Generation of Free Theorems

At `http://linux.tcs.inf.tu-dresden.de/~voigt/ft`:



Automatic Generation of Free Theorems

The theorem generated for functions of the type

```
g :: forall a . (a -> Bool) -> [a] -> [a]
```

in the sublanguage of Haskell with no bottoms is:

```
forall t1,t2 in TYPES, R in REL(t1,t2).
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall (x, y) in R. p x = q y)
   ==> (forall (z, v) in lift{[]}(R).
         (g p z, g q v) in lift{[]}(R))
```

The structural lifting occurring therein is defined as follows:

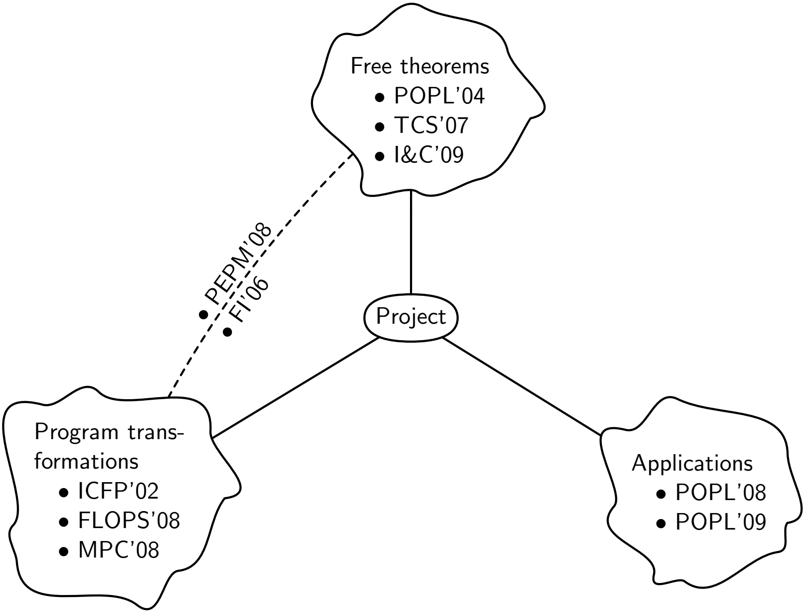
```
lift{[]}(R)
  = {([], [])}
  u {(x : xs, y : ys) |
       ((x, y) in R) && ((xs, ys) in lift{[]}(R))}
```

Reducing all permissible relation variables to functions yields:

```
forall t1,t2 in TYPES, f :: t1 -> t2.
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall x :: t1. p x = q (f x))
   ==> (forall y :: [t1]. map f (g p y) = g q (map f y))
```

Export as PDF | Show type instantiations | Enter a new type | Help page

DFG-Project VO 1512/1-1



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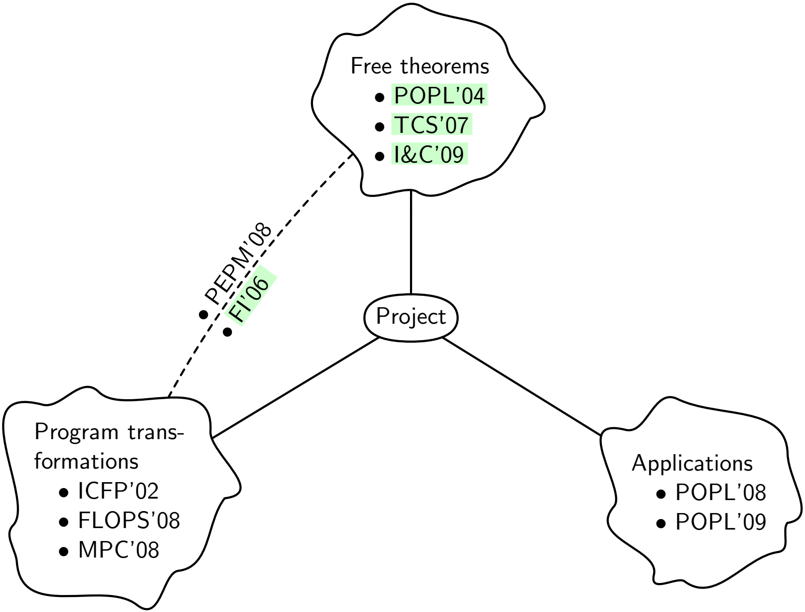

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- checking elements from l for being $\bot$
- checking p for being $\bot$
- checking outcome of p on $\bot$

... ???

Revising Free Theorems

[Wadler, FPCA'89] : for every $\texttt{g} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha]$,

$$\texttt{g}\ p\ (\texttt{map}\ f\ l) \;=\; \texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)$$

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[Johann & V., I&C'09] : taking finite failures into account

[Stenger & V., TR] : taking imprecise error semantics into account

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Better, explicit distinction. Like:

Ok v	:	nonerroneous
Bad "$\cdots$"	:	finitely failing
$\bot$	:	nonterminating

Propagation of Errors

- `tail` $[1/0, 2.5] \rightsquigarrow \mathit{Ok}\ [\mathit{Ok}\ 2.5]$

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- (`error` s_1) $+$ (`error` s_2) $\rightsquigarrow$???

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- $(\texttt{error}\ s_1) + (\texttt{error}\ s_2) \ \rightsquigarrow\ ???$

Dependence on evaluation order leads to considerably less freedom for implementors to rearrange computations, to optimise!

Imprecise Error Semantics [Peyton Jones et al., PLDI'99]

Basic idea:

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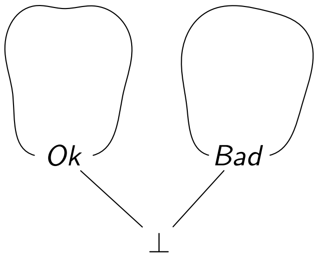

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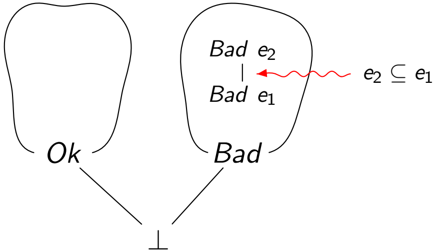
Basic idea:

$$\begin{aligned} \mathit{Ok}\ v &: \text{nonerroneous} \\ \mathit{Bad}\ \{\cdots\} &: \text{finitely failing, nondeterministic} \\ \bot &: \text{nonterminating} \end{aligned}$$

Definedness order:



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Propagation of Errors:

- $(\texttt{error}\ s_1) + (\texttt{error}\ s_2) \rightsquigarrow \mathit{Bad}\ \{s_1, s_2\}$

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- $(\texttt{error}\ s_1)\ (\texttt{error}\ s_2) \ \rightsquigarrow\ \mathit{Bad}\ \{s_1, s_2\}$
- $(\lambda x \rightarrow 3)\ (\texttt{error}\ s) \ \rightsquigarrow\ \mathit{Ok}\ 3$

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Propagation of Errors:

- $(\texttt{error}\ s_1) + (\texttt{error}\ s_2) \ \leadsto\ \mathit{Bad}\ \{s_1, s_2\}$
- $3 + (\texttt{error}\ s) \ \leadsto\ \mathit{Bad}\ \{s\}$
- $\texttt{loop} + (\texttt{error}\ s) \ \leadsto\ \bot$
- $(\texttt{error}\ s_1)\ (\texttt{error}\ s_2) \ \leadsto\ \mathit{Bad}\ \{s_1, s_2\}$
- $(\lambda x \to 3)\ (\texttt{error}\ s) \ \leadsto\ \mathit{Ok}\ 3$
- $\textbf{case}\ (\texttt{error}\ s_1)\ \textbf{of}\ \{(x, y) \to \texttt{error}\ s_2\} \ \leadsto\ \mathit{Bad}\ \{s_1, s_2\}$

Impact on Program Equivalence

"Normally":

$$\texttt{takeWhile}\ p\ (\texttt{map}\ f\ l) \quad = \quad \texttt{map}\ f\ (\texttt{takeWhile}\ (p \circ f)\ l)$$

where:

$$\begin{array}{l}
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\end{array}$$

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But now:

$$\texttt{takeWhile}\ \texttt{null}\ (\texttt{map}\ \texttt{tail}\ (\texttt{error}\ s))$$

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Because:

$$\texttt{takeWhile}\ (\texttt{null} \circ \texttt{tail})\ (\texttt{error}\ s) \leadsto \mathit{Bad}\ \{s, \text{“empty list”}\}$$

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Thus:

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Now, imagine this in the following program context:

$$\begin{aligned}&\texttt{catchJust}\ \texttt{errorCalls}\ (\texttt{evaluate}\ \cdots) \\ &\qquad (\lambda s \rightarrow \textbf{if}\ s == \text{“empty list”} \\ &\qquad\qquad \textbf{then}\ \texttt{return}\ [[42]] \\ &\qquad\qquad \textbf{else}\ \ \texttt{return}\ [\,])\end{aligned}$$

How to Revise Free Theorems?

[Wadler, FPCA'89] : for every $\texttt{g} :: (\alpha \rightarrow \mathsf{Bool}) \rightarrow [\alpha] \rightarrow [\alpha]$,

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[Johann & V., POPL'04] : in Haskell only provable if:

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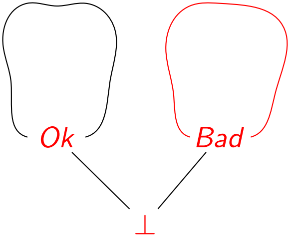
What are corresponding conditions "in real"?



Sweat and Tears ...

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(. . . similarly for "asymmetric" scenarios as well)

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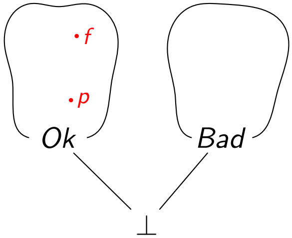

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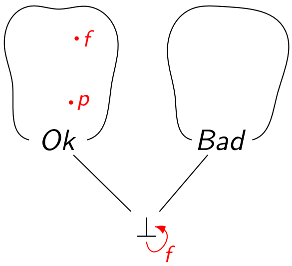

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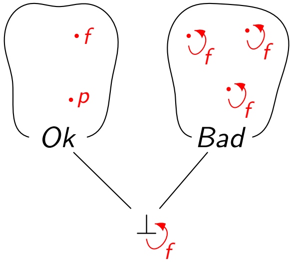

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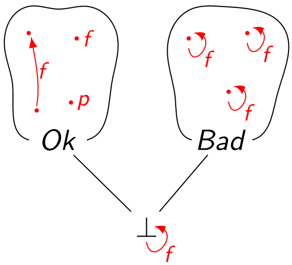
- $f\ \bot = \bot$,
- f acts as identity on erroneous values, and
- f maps nonerroneous values to nonerroneous values.



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On the practical side:

- efficiency-improving program transformations
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